JOURNAL OF APPROXIMATION THEORY **90**, 319–339 (1997)
ARTICLE NO. AT963018

Continuities of Metric Projection and Geometric Consequences*

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Communicated by Aldric L. Brown

Received March 3, 1993; accepted February 27, 1996

We discuss the geometric characterization of a subset K of a normed linear space via continuity conditions on the metric projection onto K. The geometric properties considered include convexity, tubularity, and polyhedral structure. The continuity conditions utilized include semicontinuity, generalized strong uniqueness and the non-triviality of the derived mapping. In finite-dimensional space with the uniform norm we show that convexity is equivalent to rotation-invariant almost convexity and we characterize those sets every rotation of which has continuous metric projection. We show that polyhedral structure underlies generalized strong uniqueness of the metric projection.

1. INTRODUCTION

Suppose $(X, \|\cdot\|)$ is a normed linear space and K is a closed subset of X. Let $\operatorname{dist}(x, K) := \inf\{\|x-y\| : y \in K\}$ for $x \in X$. The *metric projection* from X onto K is the set-valued mapping Π_K defined by

$$\Pi_K(x) := \{u \in K : \|x-u\| = \operatorname{dist}(x, K)\},$$

* The first author was partially supported by a sabbatical grant from Idaho State University and the research of the second author was partially supported by the AFOSR under Grant F49620-1-0045 and by the NASA/Langley Research Center under Grant NCC-1-68 Supplement-16.

0021-9045/97 $25.00

i.e., $\Pi_K(x)$ is the set of best approximations to x from K. If $\Pi_K(x) \neq \emptyset$ for all $x \in X$ we say that K is *proximinal* and if $\Pi_K(x)$ is a singleton for all $x \in X$ we say that K is a *Chebyshev* set. The characterization of the convexity of K in terms of the metric projection has attracted considerable interest ever since Bunt [11] proved that every Chebyshev subset of a finite dimensional Hilbert space must be convex. Some other contributions to this problem can be found in [25] and [26]. (See [14] for a brief survey.) An important event in the study of the convexity of Chebyshev sets was the publication of the following theorem by Vlasov [35].

THEOREM 1. *In a Banach space with rotund dual, every Chebyshev set with continuous metric projection is convex.*

Theorem 1 is an example of exploring the geometric structure of K via the continuity of the metric projection. Other developments along this line can be found in [1, 40, 38]. In particular, the assumption of continuity of the metric projection can be replaced by much weaker conditions. For example, Balaganskii proved that if K is a Chebyshev subset of a real Hilbert space and the set of discontinuities of Π_K is countable, then K is convex [1]. Even though Johnson [24] constructed a nonconvex Chebyshev set in a pre-Hilbert space (cf. also [23]), an open problem still left is whether or not a Chebyshev set in a Hilbert space is convex. Since it is well known that a closed convex set in a Hilbert space is a Chebyshev set, this open problem is related to the following conjecture on the geometric characterization of Chebyshev sets in a Hilbert space: a closed subset K of a Hilbert space X is convex if and only if K is a Chebyshev set (i.e., $\Pi_K(x)$ is a singleton for every $x \in X$). By Bunt's result [11], this conjecture is true if X is finite-dimensional. However, outside the Hilbert space setting, the focus of research was on geometric consequences of various continuity conditions on the metric projection Π_K (cf. [38]). In this paper, our goal is to characterize geometric structures of the approximation set K by using various continuity conditions on the metric projection Π_K.

The present paper is organized as follows. In Section 2, we first review some consequences of Vlasov's proof of the following result: continuity of a metric projection Π_K implies almost convexity of K. This includes a very weak continuity condition on Π_K that guarantees the almost convexity of K, as well as a characterization of a proximinal convex subset of a Banach space with a rotund dual. Then we show that, in $l_\infty(n)$ (i.e., $\mathbf{R}^n$ with the supremum norm), convexity is equivalent to rotation-invariant almost convexity. As a consequence, we characterize those sets every rotation of which has continuous metric projection, as well as rotation-invariant Chebyshev subsets of $l_\infty(n)$. In the next two sections, we will show in two ways that polyhedral structure underlies generalized strong uniqueness (or the weak sharp minimum property) of the metric projection. The main result in Section 3 is that a closed convex subset K of $l_\infty(n)$ is boundedly polyhedral if and only if the metric projection onto every rotation of K has the weak sharp minimum property. In Section 4, we prove that the unit ball of a finite-dimensional Banach space X is a polyhedron if and only if the metric projection onto every subspace of X has the weak sharp minimum property. Finally, in Section 5, we discuss possible extensions of the main results established in this paper.

We will use the following notations and conventions. Following Brown [9], we use $\Pi'_K(x)$ to denote the *derived mapping* of Π_K defined by

$$\Pi'_K(x) := \{ y \in \Pi_K(x) : x \in \operatorname{int}\{z : \Pi_K(z) \cap U \neq \varnothing\} \text{ whenever } y \in \operatorname{int}(U)\}.$$

A vector x is in $\Pi'_K(x)$ if and only if for every sequence $x_k \to x$, z is a limit point of a sequence of the form $\{w_k\}$, where $w_k \in \Pi_K(x_k)$ for every natural number k. The derived mapping can be used to characterize two continuity concepts for set-valued mappings: lower semicontinuity and almost lower semicontinuity. The metric projection Π_K is said to be *lower semicontinuous* if the set $\{x : \Pi_K(x) \cap U \neq \varnothing\}$ is open for every open set U and Π_K is said to be almost lower semicontinuous if, for every $x \in X$ and $\varepsilon > 0$.

$$\bigcup_{U \in \mathcal{O}(x)} \bigcap_{y \in U} \{\Pi_K(y) + B(0, \varepsilon)\} \neq \varnothing,$$

where $\mathcal{O}(x)$ is the collection of all open neighborhoods of x in X. When K is a proximinal subset of X, Π_K is lower semicontinuous if and only if $\Pi_K(x) = \Pi_K(x)$ [9, 10, 15]; and when $\Pi_K(x)$ is a nonempty compact set for every $x \in X$, Π_K is almost lower semicontinuous if and only if $\Pi'_K(x) \neq \varnothing$ for all $x \in X$ [15]. The metric projection Π_K is said to be *upper semicontinuous* if the set $\{x : \Pi_K(x) \cap F \neq \varnothing\}$ is closed for every closed set F and Π_K is said to be continuous if Π_K is both upper and lower semicontinuous. A continuous selection $S(\cdot)$ of Π_K is a continuous mapping from X to K such that $S(x) \in \Pi_K(x)$ for every $x \in X$. Let $B[x, \alpha]$, $B(x, \alpha)$, and $S[x, \alpha]$ stand for, respectively, the closed ball, the open ball, and the sphere with center x and radius α. When no ambiguity results, we will denote $B[0, 1]$ by B and $S[0, 1]$ by S. A closed subset A of X is called *almost convex* (or γ-sun) [36, 38] if for every closed ball $B_1 := B[x, \alpha]$ which does not intersect A and every $\alpha' > 0$ (no matter how large), there exists an $x' \in X$ such that the closed ball $B[x', \alpha']$ contains B_1 but does not intersect A. If $A \subset X$, we will denote the interior of A by $\operatorname{int}(A)$, and the boundary of A by $\operatorname{bd} A$. The space spanned by (respectively, the affine hull of) a set Y of vectors will be denoted by $\operatorname{span} Y$ (resp., $\operatorname{aff} Y$). If $x, y \in \mathbf{R}^n$, the line segment joining x and y, $\{x + \lambda(y - x) : \lambda \in [0, 1]\}$, will be denoted by $[x, y]$. By $l_\infty(n)$ we denote $(\mathbf{R}^n, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x(i)|$.

If $T: \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation represented by an orthogonal matrix with determinant equal to one, we will call T a *rotation*. Let $\mathscr{R}$ consist of all rotations. For a closed subset K of $\mathbf{R}^n$, $\mathscr{R}(K)$ will denote the collection of all rotations of K (i.e., $A \in \mathscr{R}(K)$ if and only if $A = \{Rx: x \in K\}$ for some $R \in \mathscr{R}$).

2. CONTINUITY AND CONVEXITY

In this section we first introduce a very weak continuity condition of Π_K that guarantees the almost convexity of K. As a consequence, we have a characterization of proximinal convex subsets of a Banach space with a rotund dual. Then we show that convexity is equivalent to rotation-invariant almost convexity in $l_\infty(n)$. Also we will characterize those sets which are convex and totally tubular in terms of the various continuities of the metric projection. This removes from Theorem 4.6 in [22] the prior assumption that the approximating set is convex. Finally, as a consequence, we prove that a closed subset K of $\mathbf{R}^n$ is strictly convex if and only if A is a Chebyshev subset of $l_\infty(n)$ for every $A \in \mathscr{R}(K)$.

The importance of almost convexity is its equivalence to convexity when X has a rotund dual, as shown in the following lemma by Vlasov [35, 19].

LEMMA 2 (Vlasov [35, 19]). *In a normed linear space X, every closed almost convex set is convex if and only if the dual space X^* is rotund.*

Based on Lemma 2, it suffices to show that K is an almost convex set in order to prove the convexity of K in a Banach space with a rotund dual. In [38, 19], Vlasov proved that the continuity of Π_K implies the almost convexity of K. Vlasov's proof has three clearly distinguishable steps: (i) the continuity of the metric projection Π_K implies that K is a δ-sun, (ii) every δ-sun is γ-sun, (iii) every γ-sun is almost convex [38, 19]. Recall that a nonempty closed set K is called a *δ-sun* if for any $x \notin K$ there exists a sequence $\{z_n\}$ for which $z_n \neq x$, $z_n \to x$,

$$\frac{d(z_n, K) - d(x, K)}{\|z_n - x\|} \to 1.$$

If for any $x \notin K$ and any $r > 0$ there exists a sequence $\{z_n\}$ such that $d(z_n, K) - d(x, K) \to r$, $\|z_n - x\| = r$ for all n, then K is called a *γ-sun*. However, in a Banach space, δ-suns, γ-suns, and almost convex sets are all the same [38, 19]. We observe that Vlasov's proof still holds if the continuity of Π_K is replaced by the much weaker continuity condition (1) for Π_K. The following lemma can be derived from inequality (4.9), Lemma 4.2, and Theorem 3.3 in [38] (cf. also pages 238-241 in [19]). It has also been proved in the manuscript [2]. For easy reference, we reproduce a proof by Vlasov [38, 19] here. Since we are not interested in δ-suns and γ-suns the three steps in Vlasov's proof are combined into one.

LEMMA 3 (Vlasov [38, 19]). *Suppose that for every x in the Banach space X there exists an element $p(x) \in \Pi_K(x)$ such that*

$$\lim_{\lambda \to 0^+} \operatorname{dist}(p(x), \Pi_K(x_\lambda)) = 0, \tag{1}$$

where $x_\lambda := x + \lambda(x - p(x))$. Then K is almost convex.

Proof. First we claim that for $x \in X \backslash K$ and $\lambda > 0$

$$\operatorname{dist}(x_\lambda, K) \geqslant \operatorname{dist}(x, K) + \|x_\lambda - x\| \left(1 - \frac{\|p(x) - \pi(x_\lambda)\|}{\|x - p(x)\|}\right), \tag{2}$$

where $\pi(x_\lambda)$ is any element in $\Pi_K(x_\lambda)$. To see that (2) is true, write $x = \alpha x_\lambda + (1 - \alpha)\, p(x)$, where $0 < \alpha = 1/(1 + \lambda) < 1$. Then

$$x - p(x) = \alpha(x_\lambda - p(x)) \qquad \text{and} \qquad x_\lambda - x = \frac{1 - \alpha}{\alpha}(x - p(x)) \tag{3}$$

so $\|x - p(x)\| \leqslant \|x - \pi(x_\lambda)\| \leqslant \alpha\, \|x_\lambda - \pi(x_\lambda)\| + (1 - \alpha\, \|p(x) - \pi(x_\lambda)\|$. Therefore,

$$\begin{aligned}
\operatorname{dist}(x_\lambda, K) &= \|x_\lambda - \pi(x_\lambda)\| \\
&\geqslant \frac{1}{\alpha}\, \|x - p(x)\| - \frac{1 - \alpha}{\alpha}\, \|p(x) - \pi(x_\lambda)\| \\
&= \|x_\lambda - p(x)\| - \frac{\|x_\lambda - x\|}{\|x - p(x)\|}\, \|p(x) - \pi(x_\lambda)\| \\
&= \|x - p(x)\| + \|x_\lambda - x\| \left(1 - \frac{\|p(x) - \pi(x_\lambda)\|}{\|x - p(x)\|}\right),
\end{aligned}$$

where the second equality follows from (3). This proves that (2) is true.

Now we claim that, for every $z \in X \backslash K$, $r > 0$, and $\sigma > 1$, there exists an element $x \in X$ such that

$$\operatorname{dist}(z, K) + \frac{1}{\sigma}\, \|z - x\| \leqslant \operatorname{dist}(x, K) \qquad \text{and} \qquad \|x - z\| = r. \tag{4}$$

To prove (4) we need the following Primitive Ekeland from [17] of the Bishop–Phelps Theorem [3] (cf. also page 167 in [19]): Let (Y, d) be a complete metric space and ψ be a proper but extended real lower semi-continuous function on Y bounded below. Given $\varepsilon > 0$ and $z \in Y$ there exists an $x \in Y$ such that

$$\psi(x) + \varepsilon \cdot d(x, z) \leqslant \psi(z)$$

and

$$\psi(y) > \psi(x) - \varepsilon \cdot d(x, y) \qquad \text{for all} \quad y \in Y \backslash \{x\}.$$

We apply the above Primitive Ekeland Theorem to the complete metric space $B[z, r]$ and the continuous real function ψ on $B[z, r]$ defined by

$$\psi(y) := -\operatorname{dist}(y, K).$$

For $\varepsilon = 1/\sigma$, there exists $x \in B[z, r]$ such that $\psi(x) + \varepsilon \, \|z - x\| \leqslant \psi(z)$ so

$$\operatorname{dist}(z, K) + \frac{1}{\sigma} \|z - x\| \leqslant \operatorname{dist}(x, K) \tag{5}$$

and $\psi(y) > \psi(x) - \varepsilon \, \|y - x\|$ whenever $y \neq x$ and $y \in B[z, r]$ so

$$\operatorname{dist}(y, K) < \operatorname{dist}(x, K) + \frac{1}{\sigma} \|y - x\| \text{ whenever } y \neq x \text{ and } \|y - z\| \leqslant r. \tag{6}$$

Now from (5), $\operatorname{dist}(x, K) \geqslant \operatorname{dist}(z, K) > 0$ so $x \notin K$. By (1), there exist $p(x) \in \Pi_K(x)$ and $p(x_\lambda) \in \Pi_K(x_\lambda)$ for $x_\lambda := x + \lambda(x - p(x))$ such that

$$\lim_{\lambda \to 0^+} \|p(x_\lambda) - p(x)\| = 0. \tag{7}$$

Therefore,

$$\begin{aligned} 1 &= \frac{\|x_\lambda - x\|}{\|x_\lambda - x\|} \geqslant \frac{\|x_\lambda - p(x)\| - \|x - p(x)\|}{\|x_\lambda - x\|} \geqslant \frac{\|x_\lambda - p(x_\lambda)\| - \|x - p(x)\|}{\|x_\lambda - x\|} \\ &= \frac{\operatorname{dist}(x_\lambda, K) - \operatorname{dist}(x, K)}{\|x_\lambda - x\|} \geqslant 1 - \frac{\|p(x) - p(x_\lambda)\|}{\|x - p(x)\|} \to 1, \end{aligned} \tag{8}$$

where the first inequality follows from the triangle inequality, the second inequality and the second equality are derived from the definition of $p(x_\lambda)$ and $p(x)$, the third inequality is (2), and the limit is a consequence of (7). If $\|z - x\| < r$ then $x_\lambda \in B[z, r] \backslash \{x\}$ fo $\lambda > 0$ sufficiently small. By (6) and (8),

$$\frac{1}{\sigma} \geqslant \lim_{\lambda \to 0^+} \frac{\operatorname{dist}(x_\lambda, K) - \operatorname{dist}(x, K)}{\|x_\lambda - x\|} = 1,$$

which contradicts the fact that $\sigma > 1$. Therefore, $\|z - x\| = r$. This completes the proof of (4).

Finally we are ready to prove that K is almost convex. Let $B[z, \beta]$ be a closed ball that does not intersect K. Given $\alpha > \operatorname{dist}(z, K)$, choose $\sigma > 1$ and $r > 0$ so that

$$\sigma(\alpha - \operatorname{dist}(z, K)) < r < \alpha - \beta.$$

By (4), there exists $x \in X$ such that

$$r = \|z - x\| \leqslant \sigma(\operatorname{dist}(x, K) - \operatorname{dist}(z, K))$$

Then $\operatorname{dist}(x, K) > \alpha$ so the ball $B[x, \alpha]$ does not intersect K; and $\|z - x\| < \alpha - \beta$ so $B[z, \beta] \subset B[x, \alpha]$. As a consequence, K is almost convex. ∎

For every convex proximinal subset K of X, (1) holds. As a consequence, by Lemma 3 and Lemma 2 that essentially belong to Vlasov [38, 19], we have the following characterization of convex proximinal sets in a Banach space with rotund dual.

THEOREM 4 (cf. Vlasov [38]). *Let K be a proximinal subset of a Banach space X with rotund dual. Then K is convex if and only if, for every x in X, there exists an element $p(x) \in \Pi_K(x)$ such that*

$$\lim_{\lambda \to 0^+} \operatorname{dist}(p(x), \Pi_K(x_\lambda)) = 0,$$

where $x_\lambda := x + \lambda(x - p(x))$.

Note that (1) is a much weaker condition than $\Pi'_K(x) \neq \emptyset$ for all $x \in X$. As a consequence, we have the following corollary of Lemma 3.

COROLLARY 5. *A closed subset K of a Banach space X is almost convex if one of the following conditions holds*

(i) $\Pi'_K(x) \neq \emptyset$ *for every* $x \in X$;

(ii) Π_K *has a continuous selection*;

(iii) Π_K *is lower semicontinuous*;

(iv) Π_K *is continuous.*

Remark. The condition (iv) was initially given by Vlasov (cf. Theorems 3.8 and 3.3 in [38], [6], and [36]). The condition (iii) was first introduced by Blatter [5] but he referred the reader to [37] for a proof. See also Theorems 4.15 and 3.3 in [38].

By Lemma 2 the weakest possible condition that in general guarantees the convexity of almost convex sets in a finite dimensional space is the smoothness of the norm. The convexity of a set $K \subset \mathbf{R}^n$ is defined in terms of line segments joining points in K, and so is invariant under affine mappings of $\mathbf{R}^n$ and independent of metric structure. However, the almost convexity of K depends on the shape of the unit ball in $(\mathbf{R}^n, \|\cdot\|)$ and hence it depends on the norm. To connect this fact to the failure of rotation invariance, consider in $l_\infty(2)$ the nonconvex set $K := [x, y] \cup [y, z]$, where $x := (-2, 0)$, $y := (0, 1)$, and $z := (2, 0)$. Then K is not almost convex in $l_\infty(2)$, but if K' is the image of K under a counterclockwise rotation of $\mathbf{R}^2$ through the angle $\pi/4$, then K' is almost convex in $l_\infty(2)$. Thus, almost convexity is not invariant under rotations. Another way of stating this distinction is that convexity is a "geometric" property but almost convexity is not. Our response to this problem is to explore the consequences of assuming that every rotation of a set is almost convex. The idea underlying the proof of Theorem 2 is that if there are distinct vectors $y, z \in K$ such that the open interval (y, z) does not intersect K, then there is no way to indefinitely "expand" a ball in $\mathbf{R}^n \backslash K$ (whose interior contains $(y+z)/2$) without running into y and z. The next lemma shows that there is a rotation, B', of B that has a "good side" parallel to (y, z) such that any expansion of B' must eventually contain a ray parallel to (y, z).

Lemma 6. *Suppose that $X = \mathbf{R}^n$ and $\|\cdot\| = \|\cdot\|_\infty$. Let $z = (0, ..., 0, \beta)$, where $\beta > 0$. If $0 \in B(x_j, \alpha_j)$ for every positive integer j and $\alpha_j \to \infty$, then there is a positive integer j_0 such that $\{z, -z\} \cap \operatorname{int} B(x_j, \alpha_j) \neq \emptyset$ for $j \geqslant j_0$.*

Proof. Let $x_j = (x_j^1, ..., x_j^n)$. Since $0 \in B(x_j, \alpha_j)$, we have

$$|x_j^i| < \alpha_j, \qquad \text{for} \quad i = 1, ..., n.$$

Since $\alpha_j \to \infty$, there exists j_0 such that $\alpha_j > 2\beta$ for $j \geqslant j_0$. If $x_j^n \geqslant \beta$, then $|x_j^n - \beta| \leqslant |x_j^n| < \alpha_j$ and $\|x_j - z\| < \alpha_j$. In this case, $z \in B(x_j, \alpha_j)$. If $x_j^n \leqslant \beta$, then

$$|x_j^n + \beta| < \begin{cases} 2\beta, & \text{when} \quad 0 < x_j^n, \\ \beta, & \text{when} \quad -\beta \leqslant x_j^n \leqslant 0, \\ |x_j^n|, & \text{when} \quad x_j^n < -\beta. \end{cases}$$

Thus, $|x_j^n + \beta| < \alpha_j$ and $\|x_j + z\| < \alpha_j$. In this case, $-z \in B(x_j, \alpha_j)$. ∎

Theorem 7. *Let K be a closed subset of $l_\infty(n)$. If A is almost convex for every $A \in \mathscr{R}(K)$, then K is convex.*

Proof. Suppose that K is not convex. Then we may assume with no loss of generality (rotating and translating if necessary), that there exists a vector $z := (0, ..., 0, \beta) \in K$ such that the intersection of K with the interval $[z, -z]$ is the two-point set $\{z, -z\}$. Since K is boundedly compact, there exists an $\alpha \in (0, \beta)$ such that $B[0, \alpha] \in K = \emptyset$, so Lemma 6 implies that K is not almost convex. This concludes the proof of Theorem 7. ▮

From Corollary 5(i) and Theorem 7 we obtain the following description of convexity of K by using derived mappings.

COROLLARY 8. *Let K be a closed subset of $l_\infty(n)$. If $\Pi'_A(x) \neq \emptyset$ for every $x \in \mathbf{R}^n$ and every $A \in \mathscr{R}(K)$, then K is convex.*

Remark. Since $\Pi_K(x)$ is a nonempty compact set, $\Pi'_A(x) \neq \emptyset$ for every $x \in X$ if and only if Π_A is *almost lower semicontinuous* [15]. Thus, the hypothesis "$\Pi'_A(x) \neq \emptyset$" in Theorem 8 can be replaced by "Π_A is almost lower semicontinuous."

The statement of the next result requires the definition of a *totally tubular* set. We say that $K \subset X$ is totally tubular if for all $a \in K$, $0 \neq v \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $y \in K$ and $\|a + tv - y\| < \delta$ for some $t \in \mathbf{R}$ then there exists an $s \in \mathbf{R}$ such that $y + sv \in K$ and $\|(y + sv) - a\| < \varepsilon$. Since all norms on $\mathbf{R}^n$ are topologically equivalent, total tubularity is independent of norm in $\mathbf{R}^n$. Note also that total tubularity is rotation-invariant in $\mathbf{R}^n$. Thus, total tubularity is a norm-independent and geometric concept in $\mathbf{R}^n$. For convex sets, the total tubularity is actually property (P), first introduced by Brown [8] in the study of continuity of metric projections and later extended by Wegmann [39] and Blatter, *et al.* [6]. Recall that a convex subset K of a normed linear space $(X, \|\cdot\|)$ is said to have *property* (P) (with respect to the norm $\|\cdot\|$) if for every $x \in k$, $z \in X$ with $x + z \in K$, there exist positive numbers c and δ such that $y + cz \in K$ for every $y \in K$ with $\|y - x\| < \delta$. In [22], it was shown that a closed convex subset of $\mathbf{R}^n$ is totally tubular if and only if it has property (P). The proof of Theorem 5.4 in [22] actually shows that total tubularity always implies property (P) for any closed convex subset in a normed linear space X. In fact, one can prove that property (P) also implies total tubularity for any closed convex subset in a normed linear space. The concept of total tubular convex subsets of $\mathbf{R}^n$ (or the convex sets with property (P)) was rediscovered by Huotari, Legg, and Townsend [21] in their study of convergence of the Pólya algorithm (where the term "cylindrical" was used in place of "tubular"). The terminology "total tubularity" was first used in [22], due to the fact the definition reflects the tubular structure of a set. For a closed convex set, Huotari, Legg, and Townsend's definition of total tubularity (or totally cylindrical property) was motivated by geometric intuition, while Brown's definition of total tubularity (or property (P)) was from mathematical intuition in analysis. We prefer the terminology "total tubularity," since it has a geometric meaning that fits the objective of this paper. Obviously, if K is not convex, then the total tubularity of K does not necessarily imply that K has property (P) as shown by the following example. Let

$$K := \{(x_1, x_2) \in \mathbf{R}^n : x_1^2 + x_2^2 = 1\}.$$

Then one can verify that K is totally tubular, but does not have property (P) (for $x = (0, -1)$ and $z = (0, 2)$).

Following Brown [8], we say that a norm $\|\cdot\|$ (on a linear space X) is totally tubular if its unit ball $\{x \in X \colon \|x\| \leqslant 1\}$ is a totally tubular set.

In [22] it was shown that a convex set $K \subset \mathbf{R}^n$ is totally tubular if and only if Π_A is continuous for every $A \in \mathscr{R}(K)$. This along with Theorem 8 easily establishes the following.

THEOREM 9. *Suppose K is a closed subset of $l_\infty(n)$. Then K is a totally tubular convex set if and only if for every $A \in \mathscr{R}(K)$, Π_A is continuous.*

Note that if K is a closed and strictly convex subset of $\mathbf{R}^n$, then K is a Chebyshev subset of $l_\infty(n)$. It is well-known that the metric projection Π_K is continuous if K is a Chebyshev subset of a finite-dimensional Banach space. As a consequence of Theorem 9, if A is a Chebyshev subset of $l_\infty(n)$ for every $A \in \mathscr{R}(K)$, then K is convex. If bd$K$ contains a nontrivial line segment, say $[a, b]$ with $a \neq b$, then there is a supporting hyperplane H of K that contains $[a, b]$. Let u be a unit vector normal to H such that $\{a + \lambda u \colon 0 < \lambda \leqslant 1\} \cap K = \varnothing$. Let R be a rotation such that $R(u) = (1, 0, ..., 0)$. Then $R(H)$ is parallel to the coordinate plane $\{x \colon x(1) = 0\}$ and $\Pi_{R(K)}(z) \supset R([a, b])$ for $z := R(a) + (\|R(a-b)\|_\infty, 0, ..., 0)$. This is impossible, since $R(K)$ is a Chebyshev set. Therefore, if A is a Chebyshev subset of $l_\infty(n)$ for every $A \in \mathscr{R}(K)$, then K is strictly convex. This provides a metric characterization of strict convexity, and a geometric characterization of rotation-invariant Chebyshev subsets of $l_\infty(n)$.

THEOREM 10. *Suppose that K is a closed subset of $l_\infty(n)$. Then K is strictly convex if and only if A is a Chebyshev subset of $l_\infty(n)$ for every $A \in \mathscr{R}(K)$.*

3. WEAK SHARP MINIMA AND LOCAL POLYHEDRAL STRUCTURE

A best approximation is, loosely speaking, *strongly unique* if the distance from the approximatee to elements of the approximating set, in every ray emanating from the best approximation, grows at a linear rate. This growth-of-distance notion is easily generalized to the context where there may be more than one best approximation. In this section we will show that local polyhedral structure underlies the linear growth of distance to a fixed approximatee.

Suppose $\|\cdot\|$ is any norm on $\mathbf{R}^n$. If $x \in \mathbf{R}^n$ is fixed, define a function $F_x: \mathbf{R}^n \to \mathbf{R}$ by $F_x(y) := \|x - y\|$. Suppose that $K \subset \mathbf{R}^n$ is closed and convex. Note that $\Pi_K(x)$ is the convex set of minima of the function F_x restricted to K. We say that $\Pi_K(x)$ is a set of *weak sharp minima* for F_x relative to K if there is an $\alpha > 0$ such that

$$\|x - y\| \geqslant \operatorname{dist}(x, K) + \alpha \cdot \operatorname{dist}(y, \Pi_K(x))$$

for every $y \in K$ (cf. [12, 13, 16, 18, 20, 28, 29, 30, 31, 32, 33] for some related research on strong uniqueness and weak sharp minima). We say that the metric projection Π_K has the *weak sharp minimum property* if $\Pi_K(x)$ is a set of weak sharp minima for every $x \in \mathbf{R}^n$. Note that if $\Pi_K(x)$ is a singleton then it is a set of weak sharp minima for F_x relative to K if and only if its one element is the *strongly unique* best $\|\cdot\|$ approximation to x from K. Thus the weak sharp minimum property can be considered as a kind of generalized strong uniqueness. See [13] for an introduction to strong unicity. Note that by Cheney's argument [13] if $\Pi_K(x)$ is a set of weak sharp minima for every $x \in \mathbf{R}^n$ with α independent of x, then Π_K is Lipschitz continuous (see also [31, 33]).

A subset of $\mathbf{R}^n$ is called a *polyhedron* if it is the intersection of a finite number of closed half-spaces. A set is called *boundedly polyhedral* if its intersection with every bounded polyhedron is a polyhedron. In the discussion of our first polyhedral-structure result we will abuse notation by writing $D(A)$ when we mean $\{Dv: v \in A\}$, where D is an $n \times n$ matrix and $A \subset \mathbf{R}^n$. We begin with two preparatory lemmas.

LEMMA 11. *Let K be a subset of $l_\infty(n)$. If Π_A has the weak sharp minimum property for all $A \in \mathscr{R}(K)$, then Π_B also has the weak sharp minimum property for $B \in \mathscr{Q}(K)$, where $\mathscr{Q}(K) := \{Q(K) - z$: Q is an orthogonal matrix and $z \in R^n\}$.*

Proof. Let $B = Q(K) - z \in \mathscr{Q}(K)$. we consider the two cases: $\det(Q) = \pm 1$.

Suppose $\det(Q) = 1$. Then Q is a rotation. Let $A := Q(K)$. Then it is easy to verify that

$$\begin{aligned} &\operatorname{dist}(x, B) = \operatorname{dist}(x + z, A), \\ &\Pi_B(x) = \Pi_A(x + z) - z, \qquad (9) \\ &\operatorname{dist}(w, \Pi_B(x)) = \operatorname{dist}(w + z, \Pi_A(x + z)). \end{aligned}$$

For every $w \in B$, we have $w+z \in A$. Therefore by the weak sharp minimum property of Π_A there exists a positive constant γ such that, for every $w \in B$,

$$\begin{aligned} \|x-w\| &= \|(x+z)-(w+z)\| \\ &\geqslant \operatorname{dist}(x+z, A)+\gamma \cdot \operatorname{dist}(w+z, \Pi_A(x+z)). \end{aligned} \tag{10}$$

It follows from (9) and (10) that $\|x-w\| \geqslant \operatorname{dist}(x, B)+\gamma \cdot \operatorname{dist}(w, \Pi_B(x))$. That is, Π_B also has the weak sharp minimum property.

Suppose $\det(Q)=-1$. Let D be the diagonal matrix whose ith diagonal entry is -1 for $i=1$ and 1 for $i \geqslant 2$. Then D represents the reflection about the hyperplane $x(1)=0$. Since DQ is an orthogonal matrix with determinant 1, DQ is a rotation. Let $\bar{B}:=D(B) \equiv DQ(K)-Dz$. By the proof in the case $\det(Q)=1$, $\Pi_{\bar{B}}$ has the weak sharp minimum property. Since the l_∞ distance is invariant under the reflection D, one can easily verify that

$$\begin{aligned} &\|Dx\| = \|x\|, \\ &\operatorname{dist}(x, B) = \operatorname{dist}(Dx, \bar{B}), \\ &D\Pi_B(x) = \Pi_{\bar{B}}(Dx), \\ &\operatorname{dist}(w, \Pi_B(x)) = \operatorname{dist}(Dw, \Pi_{\bar{B}}(x)). \end{aligned} \tag{11}$$

If $w \in B$ then $Dw \in \bar{B}$. By the weak sharp minimum property of $\Pi_{\bar{B}}$, there exists a positive constant γ such that

$$\begin{aligned} \|x-w\| &= \|Dx-Dw\| \\ &\geqslant \operatorname{dist}(Dx, \bar{B})+\gamma \cdot \operatorname{dist}(Dw, \Pi_{\bar{B}}(x)) \qquad \text{for} \quad w \in B. \end{aligned} \tag{12}$$

Combining (11) and (12), we obtain the weak sharp minimum property of Π_B:

$$\|x-w\| \geqslant \operatorname{dist}(x, B)+\gamma \cdot \operatorname{dist}(w, \Pi_B(x)) \qquad \text{for} \quad w \in B.$$

This concludes the proof of Lemma 11. ∎

LEMMA 12. *Let $\|\cdot\|$ be any norm on $\mathbf{R}^n$. If a convex set K is not boundedly polyhedral, then there exist $z \in K$, $u \in S[0, 1]$ and a hyperplane H which supports K at z such that*

(i) $z+tu \in H$ *for all* t;

(ii) $z+tu \notin K$ *for* $t>0$;

(iii) *for each positive integer* k *there exist* $v^k \in K$ *and* $0 < t_k < 1$ *such that*

$$\lim_{k \to \infty} \frac{\|v^k - (z + t_k u)\|}{t_k} = 0.$$

Proof. We may suppose without loss of generality that 0 is in the relative interior of K. By Theorem 4.7 in [27] there is a two dimensional subspace P such that $K^* := K \cap P$ is not boundedly polyhedral. Thus, K^* has a sequence $\{v^k\}$ of distinct extreme points which converges to, say, z. For each natural number k, let $t_k := \|v^k - z\|$ and $u_k := (v^k - z)/t_k$. We may assume without loss of generality that there exists a $u \in S[0, 1]$ such that $\lim_{k \to \infty} \|u^k - u\| = 0$. Let L denote the line containing the vectors $\{z + tu : t \in \mathbf{R}\}$. By the Separation Theorem, (11.3) in [34], there exists a hyperplane H which contains L and supports K at z. It is easy to see that (i), (ii) and (iii) hold. This concludes the proof of Lemma 12. ∎

THEOREM 13. *Suppose that* K *is a closed convex subset of* $l_\infty(n)$. *Then the following are equivalent*:

(i) *For every* $A \in \mathscr{R}(K)$, Π_A *has the weak sharp minimum property.*

(ii) *The set* K *is a convex boundedly polyhedral set.*

Proof. Suppose (i) holds. If K is not boundedly polyhedral, then there exist z, u, v_k, t_k and H satisfying (i), (ii) and (iii) of Lemma 12. Obviously there exists an orthogonal matrix Q such that $Q(H - z) = \{x : x(n) = 0\}$, $Qu = e_1 := (1, 0, ..., 0)$ and $\bar{K} := Q(K - z) \equiv Q(K) - Qz \subset \{x : x(n) \leqslant 0\}$. By Lemma 11, $\Pi_{\bar{K}}$ has the weak sharp minimum property. Therefore, after replacing K by $\bar{K}$, we may assume that

$$\begin{aligned} &\Pi_K \text{ has the weak sharp minimum property;} \\ &0 \in K \subset \{x : x(n) \leqslant 0\}; \\ &te_1 \notin K \text{ for } t > 0; \\ &\text{there exist } v_k \in K \text{ and } 0 < t_k < 1 \text{ such that } \lim_{k \to \infty} \frac{\|v^k - t_k e_1\|}{t_k} = 0. \end{aligned} \tag{13}$$

Let $K_0 = K \cap \{x : x(n) = 0\}$. Since $e_1 \notin K_0$, in the subspace $\{x : x(n) = 0\}$ there exists a hyperplane H_0 which supports K_0 at 0 and separates the line segment $[0, e_1]$ from K_0. That is, there exist $a(1), ..., a(n-1)$ such that

$$K_0 \subset \{x : a(1)\, x(1) + \cdots + a(n-1)\, x(n-1) \leqslant 0\} \tag{14}$$

and

$$e_1 \in \{x: a(1)\, x(1) + \cdots + a(n-1)\, x(n-1) \geqslant 0\}. \tag{15}$$

Note that (15) is equivalent to $a(1) \geqslant 0$.

If $(a(2), ..., a(n-1)) \neq 0$, let

$$\beta := \sqrt{\frac{1}{n-2} \sum_{i=2}^{n-1} [a(i)]^2}$$

and let Q be any orthogonal matrix such that $Q(0, a(2), ..., a(n-1), 0) = \beta(0, 1, ..., 1, 0)$ and $(Qx)(i) = x(i)$ for $i=1$ or $i=n$. (Q represents an orthogonal transformation in $(0, x(2), ..., x(n-1), 0)$-space.) Then, after replacing K by $Q(K)$, (13) still holds; but (14) and (15) become the following condition:

$$K_0 \subset \{x: \alpha x(1) + \beta(x(2) + \cdots + x(n-1)) \leqslant 0\}, \tag{16}$$

where $\alpha := a(1) \geqslant 0$. Note that $\beta \geqslant 0$. If $(a(2), ..., a(n-1)) = 0$, then (16) holds with $\alpha = a(1) \geqslant 0$ and $\beta = 0$.

Now let $x(i) = 1$ for $1 \leqslant i \leqslant n$. Then we claim that

$$\operatorname{dist}(x, K) = 1 \quad \text{and} \quad v^*(1) = 0 \qquad \text{for every} \quad v^* \in \Pi_K(x). \tag{17}$$

In fact for every $v^* \in \Pi_K(x)$

$$1 = \|x\| = \|x - 0\| \geqslant \|x - v^*\| = \max_{1 \leqslant i \leqslant n} |1 - v^*(i)| \geqslant \max_{1 \leqslant i \leqslant n} (1 - v^*(i)) \geqslant 1,$$

since $0 \in K$ and $v(n) \leqslant 0$ for every $v \in K$. Since $\operatorname{dist}(x, K) = 1$ and $0 \in K$,

$$v^*(i) \geqslant 0 \qquad \text{for every} \quad 1 \leqslant i \leqslant n-1. \tag{18}$$

Since $\Pi_K(x) \subset K_0$ and $\beta \geqslant 0$, we have

$$\alpha v^*(1) \leqslant \alpha v^*(1) + \beta(v^*(2) + \cdots + v^*(n-1)) \leqslant 0 \quad \text{for } v^* \in \Pi_K(x). \tag{19}$$

If $\alpha > 0$, by (18) and (19), $v^*(1) = 0$ for every $v^* \in \Pi_K(x)$. Otherwise $\alpha = 0$ and $\beta > 0$. By (18) and (19) we have $v^*(i) = 0$ for $2 \leqslant i \leqslant n-1$. Therefore $v^* = v^*(1)\, e_1$. But $te_1 \notin K_0$ for every $t > 0$. Hence we have $v^*(1) = 0$ in this case also. This proves (17).

Finally, we are ready to derive the contradiction. By the weak sharp minimum property of Π_K there exists a positive constant γ such that

$$\|x - v_k\| \geqslant \operatorname{dist}(x, K) + \gamma \cdot \operatorname{dist}(v_k, \Pi_K(x)). \tag{20}$$

However,

$$\begin{aligned} \|x - v_k\| &\leqslant \|x - t_k e_1\| + \|v_k - t_k e_1\| \\ &= 1 + \|v_k - t_k e_1\| = \operatorname{dist}(x, K) + \|v_k - t_k e_1\| \end{aligned} \tag{21}$$

and by the triangle inequality and the definition of dist$(\cdot, \cdot)$

$$\begin{aligned} \operatorname{dist}(v_k, \Pi_K(x)) &\geqslant \operatorname{dist}(t_k e_1, \Pi_K(x)) - \|v_k - t_k e_1\| \\ &\geqslant \min_{v^* \in \Pi_K(x)} |t_k - v^*(1)| - \|v_k - t_k e_1\| \\ &= t_k - \|v_k - t_k e_1\|. \end{aligned} \tag{22}$$

It follows from (20), (21), (22) and (13) that

$$\gamma \leqslant \frac{\|x - v_k\| - \operatorname{dist}(x, K)}{\operatorname{dist}(v_k, \Pi_K(x))} \leqslant \frac{\|v_k - t_k e_1\|}{t_k - \|v_k - t_k e_1\|} = \frac{\|v_k - t_k e_1\|/t_k}{1 - \|v_k - t_k e_1\|/t_k} \to 0.$$

which is impossible. This contradiction shows that (i) implies (ii).

Conversely let $x \in \mathbf{R}^n$. If $x \notin K$, define

$$g(y) := \frac{\|x - y\| - \operatorname{dist}(x, K)}{\operatorname{dist}(y, \Pi_K(x))}.$$

Then $g(y)$ is a continuous function on $\mathbf{R}^n \backslash \Pi_K(x)$. Since $\Pi_K(x)$ is a bounded set and $\lim_{\|y\| \to \infty} g(y) = 1$, there exists a positive constant λ such that $\Pi_K(x) \subset \{ y \in \mathbf{R}^n : \|y\|_\infty \leqslant \lambda\}$ and $g(y) \geqslant 1/2$ whenever $\|y\|_\infty \geqslant \lambda$. Thus,

$$\|x - y\| - \operatorname{dist}(x, K) \geqslant \tfrac{1}{2} \operatorname{dist}(y, \Pi_K(x)) \qquad \text{whenever} \quad \|y\|_\infty \geqslant \lambda. \tag{23}$$

If $x \in K$, then we choose λ such that $\Pi_K(x) \subset \{ y \in \mathbf{R}^n : \|y\|_\infty \leqslant \lambda\}$. In this case, (23) also holds.

By the definition of convex boundedly polyhedral sets, $\bar{K} := \{ y \in K : \|y\|_\infty \leqslant \lambda\}$ is a convex polyhedral set. It was proven in [31] that if $\bar{K}$ is polyhedral and $\|\cdot\|$ is a polyhedral norm then there exists a positive constant γ such that

$$\|x - y\| \geqslant \operatorname{dist}(x, \bar{K}) + \gamma \cdot \operatorname{dist}(y, \Pi_{\bar{K}}(x)) \qquad \text{for every} \quad y \in \bar{K}.$$

It is easy to verify that $\Pi_{\bar{K}}(x) = \Pi_K(x)$ and $\operatorname{dist}(x, K) = \operatorname{dist}(x, \bar{K})$, since $\Pi_K(x) \subset \bar{K} \subset K$. Thus we actually have

$$\|x - y\| \geqslant \operatorname{dist}(x, K) + \gamma \cdot \operatorname{dist}(y, \Pi_K(x)) \qquad \text{for every} \quad y \in \bar{K}.$$

This along with (23) proves that Π_K has the weak sharp minimum property. ∎

Remark. Let K be the convex hull of $\{(n, n^2): n = 0, 1, ...\} \subset \mathbf{R}^2$. Then K is not a polyhedral set, but K is a boundedly polyhedral set. Therefore, by Theorem 13, Π_K has the weak sharp minimum property. The weak sharp minimum property of Π_A for $A \in \mathscr{R}(K)$ is characterized by the local polyhedral structure of K.

4. WEAK SHARP MINIMA AND POLYHEDRAL NORM

We hope that the following theorem will contribute to the discussion of approximation theoretic characterizations of geometric properties of the norm itself. Recall Lemma 2, which characterizes smooth norms for reflexive spaces. Björnestal [4] showed that in a uniformly convex Banach space the order of strong unicity of the metric projection operator can be written in terms of the inverse of the modulus of convexity. Brown [8] proved that the metric projection onto every finite-dimensional subspace of a normed linear space X is continuous if and only if X has a so-called (P)-norm (i.e., the unit ball of X has property (P)). Note that a set K in a finite-dimensional space has property (P) if and only if K is totally tubular [22]. Therefore, when X is finite-dimensional, Brown's result can be restated as follows. The metric projection onto every subspace of a finite-dimensional normed linear space X is continuous if and only if the unit ball of X is totally tubular. Huotari and Sahab [20] showed that in certain cases the modulus of convexity of the norm is characterized in terms of the order of strong unicity of the metric projection. All these results show that there is a connection between the various continuity conditions of metric projections and the geometric characteristics (or the "shape") of the unit ball in a normed linear space. This raises the question, which we now answer, about the geometric consequences of assuming that Π_K has the weak sharp minimum property for every subspace K of a finite-dimensional normed linear space $(\mathbf{R}^n, \|\cdot\|)$.

THEOREM 14. *For any given norm $\|\cdot\|$ on $\mathbf{R}^n$, the following are equivalent.*

(i) *The norm $\|\cdot\|$ is polyhedral (i.e., the unit ball in $(\mathbf{R}^n, \|\cdot\|)$ is a polyhedron).*

(ii) *For every subspace V of $\mathbf{R}^n$, Π_V has the weak sharp minimum property.*

(iii) *For every one-dimensional subspace L of $\mathbf{R}^n$, Π_L has the weak sharp minimum property.*

Proof. The implication that (i) $\Rightarrow$ (ii) was proved by Deutsch and Li [16] (cf. also [31]). The implication that (ii) $\Rightarrow$ (iii) is trivial. Now we prove that (iii) $\Rightarrow$ (i) by contradiction.

Suppose that $\|\cdot\|$ is not polyhedral. Then the unit ball B is not boundedly polyhedral. Applying Lemma 12 to $K \equiv B$ we obtain that there exist v_k, $z \in B$, a unit vector u and a hyperplane H which supports B at z such that conditions (i), (ii) and (iii) in Lemma 12 hold. Let $L := \{tu\colon t \in \mathbf{R}\}$. Then L is a one-dimensional subspace of $\mathbf{R}^n$. We claim that Π_L does not have the weak sharp minimum property at z.

For every t, $z - tu \in H$ and H supports B at z. Thus $\|z - tu\| \geqslant 1$ for $t \in \mathbf{R}$ and $\operatorname{dist}(z, L) = 1$. For $t < 0$, $z - tu \in H \backslash B$; i.e., $\|z - tu\| > 1$. Therefore $\Pi_L(z) \subset \{tu\colon t \geqslant 0\}$. Let $y_k := -t_k u \in L$. Then

$$\begin{aligned} \operatorname{dist}(y_k, \Pi_L(z)) &= \inf\{\|y_k - tu\| : tu \in \Pi_L(z)\} \\ &\geqslant \inf\{\|-t_k u - tu\| : t \geqslant 0\} = t_k. \end{aligned} \tag{24}$$

On the other hand,

$$\|z - y_k\| = \|z + t_k u\| \leqslant \|v_k\| + \|v_k - (z + t_k u)\| \leqslant 1 + \|v_k - (z + t_k u)\|. \tag{25}$$

It follows from the fact that $\operatorname{dist}(z, L) = 1$, (24), (25) and (iii) of Lemma 12 that

$$\frac{\|z - y_k\| - \operatorname{dist}(z, L)}{\operatorname{dist}(y_k, \Pi_L(z))} \leqslant \frac{\|v_k - (z + t_k u)\|}{t_k} \to 0.$$

So Π_L does not have the weak sharp minimum property at z. This contradiction proves Theorem 14. ∎

5. CONJECTURES AND EXAMPLES

After reading the first submitted version of this paper, the editor A. L. Brown pointed out: "The 2-norm $\|\cdot\|_2$ is invariant with respect to every orthogonal transformation of $\mathbf{R}^n$, but given another norm the orthogonal transformations would seem to represent an arbitrary choice. If $S\colon \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism (corresponding to a change of basis) and T is orthogonal then $S^{-1}TS$ is equally a candidate for a rotation." In fact, geometric characteristics such as convexity, total tubularity, and the polyhedral property are invariant under affine transformations. For example, if $K \subset \mathbf{R}^n$ is a convex (resp., totally tubular, polyhedral, boundedly polyhedral) set, so is its affine image $Q(K) := \{Q(x)\colon x \in K\}$, where Q is an affine mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$. Let $\mathscr{A}$ be the collection of all affine mappings from $\mathbf{R}^n$ to

$\mathbf{R}^n$ and $\mathscr{A}(K)$ be the collection of all affine images of K (i.e., $A \in \mathscr{A}(K)$ if and only if $A = Q(K)$ for some $Q \in \mathscr{A}$). Then Theorem 9 and Theorem 13 still hold if we replace $\mathscr{R}(K)$ by any collection $\mathscr{D}$ such that $\mathscr{R}(K) \subset \mathscr{D} \subset \mathscr{A}(K)$. The emphasis of Theorem 9 and Theorem 13 is on the relationship between the rotation-invariant analytic behavior of K and its geometric consequences. We believe that the main results in this paper can be generalized as follows.

CONJECTURE 15. Let K be a closed subset of $(\mathbf{R}^n, \|\cdot\|)$.

(i) If Π_A is almost lower semicontinuous for every $A \in \mathscr{R}(K)$, then K is convex.

(ii) Suppose that $\|\cdot\|$ is a totally tubular norm. Then Π_A is continuous for every $A \in \mathscr{R}(K)$ if and only if K is a totally tubular convex set.

(iii) Suppose that $\|\cdot\|$ is a polyhedral norm (i.e., the unit ball is a polyhedral set) and K is convex. Then Π_A has the weak sharp minimum property for every $A \in \mathscr{R}(K)$ if and only if K is a convex boundedly polyhedral set.

(iv) Suppose that K is convex and is not a singleton. The metric projection Π_A has the weak sharp minimum property for every $A \in \mathscr{R}(K)$ if and only if $\|\cdot\|$ is a polyhedral norm and K is a convex boundedly polyhedral set.

Our study leads us to believe that in $(\mathbf{R}^n, \|\cdot\|_1)$ a rotation-invariant almost convex set might not be convex. (This issue will be addressed in a future publication.) However the almost lower semicontinuity of Π_A for every $A \in \mathscr{R}(K)$ might imply that A is a sun for every $A \in \mathscr{R}(K)$, which in turn might imply the convexity of K. (See [7] for the definition of sun.) The foundation for Conjecture 15(ii) is the fact that if $\|\cdot\|$ is a totally tubular norm and K is a convex totally tubular set then Π_A is continuous for every $A \in \mathscr{R}(K)$ (cf. [22]). It seems that our proof of Theorem 13 can be modified to prove the "only if" part of Conjecture 15(iii).

Brown proved that if Π_K is continuous for any subspace of $\mathbf{R}^n$ then $\|\cdot\|$ is a totally tubular norm [8, 22]. Therefore the totally tubular norm assumption in Conjecture 15(ii) is necessary. Theorem 14 shows that the polyhedral norm assumption in Conjecture 15(iii) is also necessary. Note that Π_K is a contraction (i.e., Lipschitz continuous with Lipschitz constant 1) for any closed convex subset K of a Hilbert space. Thus, the weak sharp minimum property in Conjecture 15(iii) can not be weakened to the Lipschitz continuity of Π_K.

We conclude the paper with an example which shows that the condition in Conjecture 15(ii) can't be generalized to the existence of continuous selections. We believe this example is well-known and may have been given before without proof.

EXAMPLE 16. There is a closed convex non-totally tubular subset K of $l_\infty(3)$ such that Π_A has a continuous metric selection for every $A \in \mathscr{R}(K)$.

Proof. Let $C := \{(x(1), x(2), x(3)) : x(1)^2 + (x(2)-1)^2 = 1, x(3) = 0\}$, $P := (0, 1, 1)$, and $Q := (0, 0, -2)$; and let K be the convex hull of $C \cup \{P\} \cup \{Q\}$. To see that K is not totally tubular let $a := Q$, $v := (0, 0, 1)$, and $y_k := (\sqrt{1-(1/k-1)^2}, 1/k, 0)$.

Since $L := [(0, 0, -2), (0, 0, 0)]$ is the only line segment in bd$K$ which is parallel to a face of B, it must be that $\Pi_K(x)$ is either a singleton or a closed subinterval of L. If γ is a selection for Π_K let $\gamma_i(x)$ be the i-th coordinate of $\gamma(x)$, $i = 1, 2, 3$. Define $\gamma: \mathbf{R}^3 \to K$ by requiring that $\gamma(x)$ be the element of $\Pi_K(x)$ with $\gamma_3(x) \geqslant y(3)$, $y \in \Pi_K(x)$. If $\Pi_K(x) = \{\gamma(x)\}$, then clearly γ is continuous at x, so we suppose that $\Pi_K(x) \subset L$.

We will show that γ is continuous at x by analyzing sequences converging to x. Let $\{x_j\}$ be a sequence in $\mathbf{R}^3$ converging to x. Let y^* be a limiting point of $\{\gamma(x_j)\}$. By selecting a subsequence, we may assume that $\lim_j \gamma(x_j) = y^*$.

Let $\delta_j := \operatorname{dist}(x_j, K)$ and $\delta := \operatorname{dist}(x, K)$. Since $\operatorname{dist}(z, K)$ is a continuous function of z, $\delta_j \to \delta$ and $y^* \in \Pi_K(x)$. Thus

$$B[x_j, \delta_j] \to B[x, \delta] \tag{26}$$

and $y^*(i) = \gamma_i(x)$, $i = 1, 2$.

Suppose $J := \{j : \Pi_K(x_j) \subset L\}$ contains infinitely many indices. Then $\Pi_K(x_j) = \Pi_L(x_j)$ for $j \in J$ and it follows from Theorem 9 that

$$\lim_{j \in J} \Pi_K(x_j) = \lim_{j \in J} \Pi_L(x_j) = \Pi_L(x) = \Pi_K(x).$$

Thus $\gamma_3(x_j) \to \gamma_3(x)$ and we are done.

Since $y^* \in \Pi_K(x) \subset L$, $y^*(3) \leqslant 0$. If $y^*(3) = 0$, by the definition of γ, we have $0 \geqslant \gamma_3(x) \geqslant y^*(3) = 0$. As a consequence, $\gamma_3(x) = 0 = y^*(3)$ and $\lim_j \gamma(x_j) = \gamma(x)$.

Now suppose $\Pi_K(x_j) \cap L = \varnothing$ for j large enough and $y^*(3) < 0$. Then $\gamma_3(x_j) < 0$ for j large enough. Thus, there exists a positive integer j^* such that $\Pi(x_j) = \{\gamma(x_j)\}$ is a singleton and $\gamma_3(x_j) < 0$ for $j \geqslant j^*$. Then $B(x_j, \delta_j)$ contains a point with third coordinate smaller than that of a point in K. From the geometric construction of K, $\gamma_3(x_j) \geqslant y(3)$ for $y \in B[x_j, \delta_j]$. By (26) $\gamma_3(x) \leqslant 0$ and $\gamma_3(x_j) \to \gamma_3(x)$.

This proves that $\gamma(\cdot)$ is a continuous selection for Π_K. If $A = R(K) \in \mathscr{R}(K)$ and bd$A$ contains a line segment parallel to a face of B then we may construct a continuous selection for Π_A as above by choosing the element of $\Pi_K(x)$ nearest to $R(C)$. ∎

ACKNOWLEDGMENTS

Part of this work was done while the first author was visiting Old Dominion University. He is indebted to the Department of Mathematics and Statistics for its support during that time. Substantial improvements in this paper were made possible by helpful suggestions from the referees and from the editor, A. L. Brown.

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